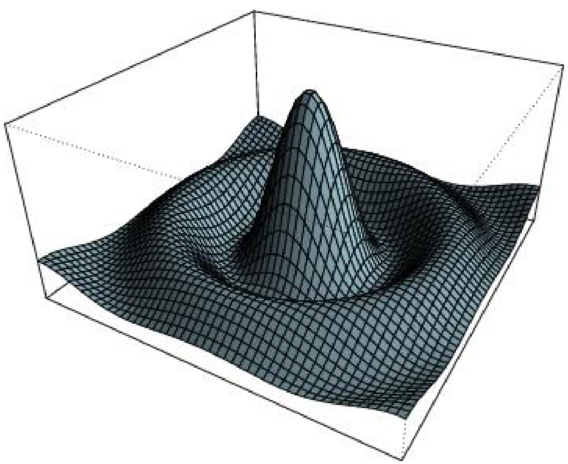
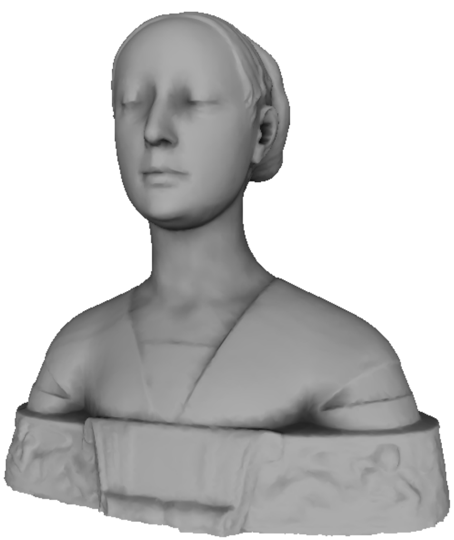
Data structures for 3D Meshes

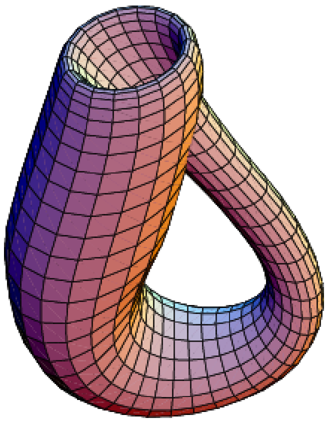
Paolo Cignoni
p.cignoni@isti.cnr.it
http://vcg.isti.cnr.it/~cignoni

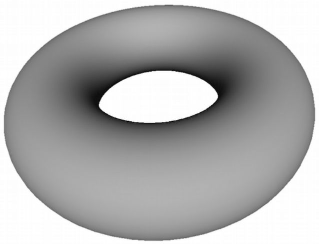
Surfaces

- A 2-dimensional region of 3D space
- *A portion of space having length and breadth but no thickness*







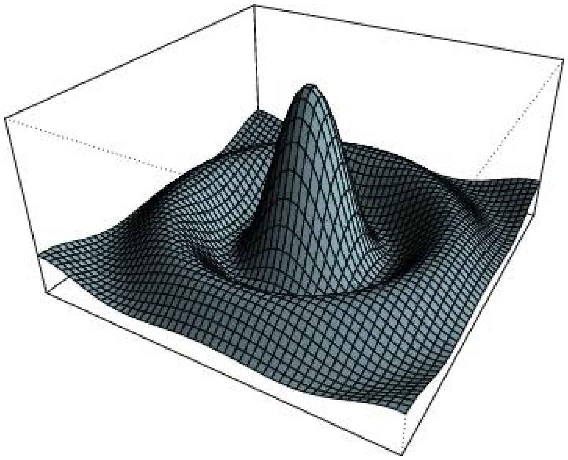


Defining Surfaces

- Analytically...
  - Parametric surfaces

$$S: R^2 \rightarrow R^3$$

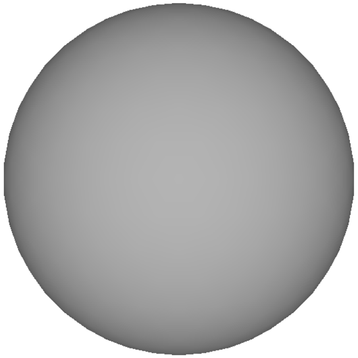
$$S(x,y) = \left(x, y, \sin\left(\sqrt{(x^2+y^2)}\right)/\sqrt{(x^2+y^2)}\right)$$



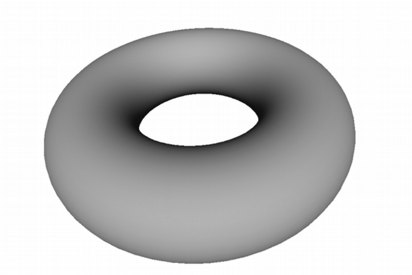
  - Implicit surfaces

$$S = \{(x,y,z): f(x,y,z) = 0\}$$

$$S = \{(x,y,z): x^2 + y^2 + z^2 - r^2 = 0\}$$



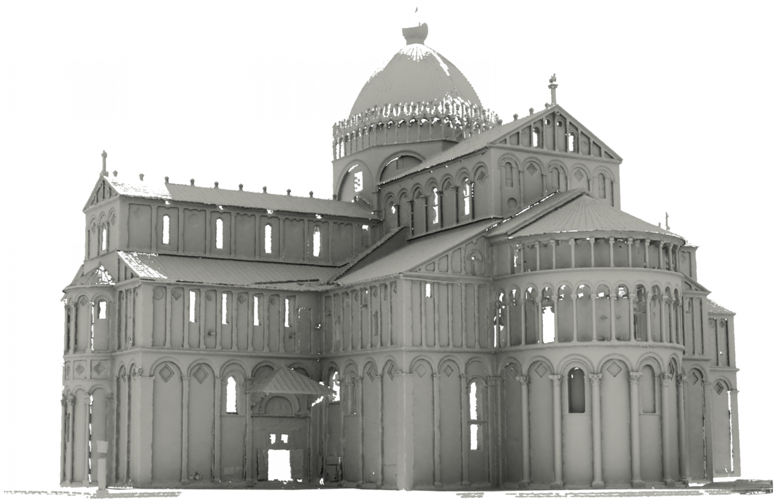
$$S = \{(x,y,z): (x^2 + y^2 + R^2 - r^2)^2 - 4R^2(x^2+y^2) = 0\}$$



Representing Real World Surfaces

- Analytic definition falls short of representing *real world* surfaces in a *tractable* way

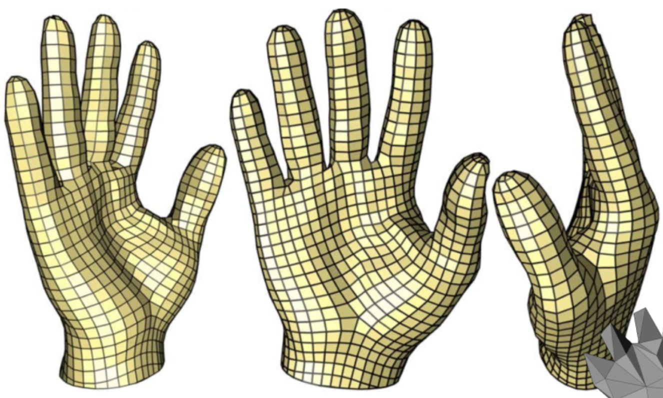
$$S(x,y) = ...?$$



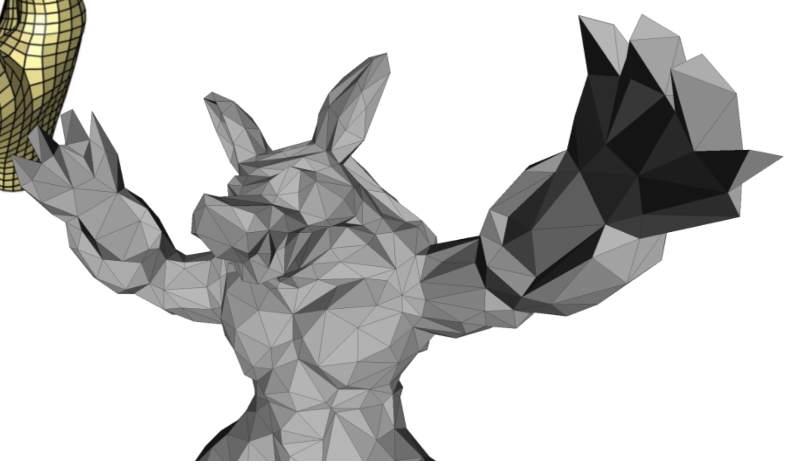
... surfaces can be represented by ***cell complexes***

Cell complexes (meshes)

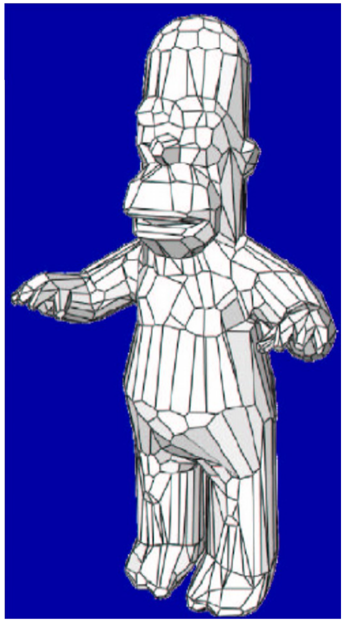
- Intuitive description: a continuous surface divided in polygons



quadrilaterals (quads)



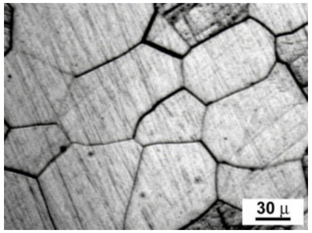
triangles

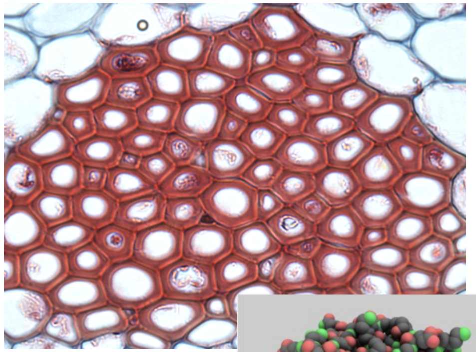
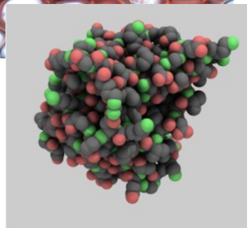


Generic polygons

Cell Complexes (meshes)

- In nature, meshes arise in a variety of contexts:
  - Cells in organic tissues
  - Crystals
  - Molecules
  - …
  - Mostly *convex* but *irregular* cells
  - Common concept: *complex* shapes can be described as *collections* of *simple building blocks*

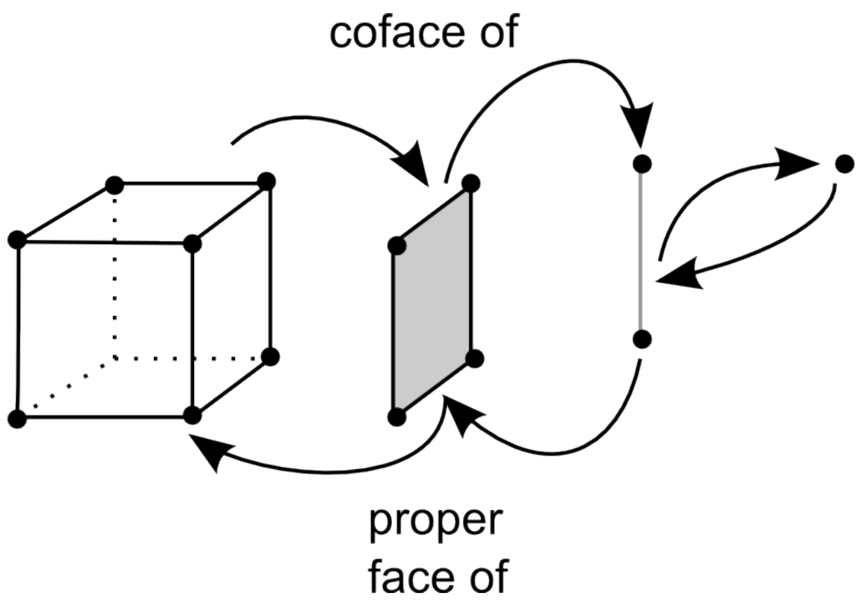


6

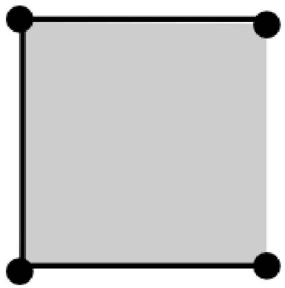
Cell Complexes (meshes)

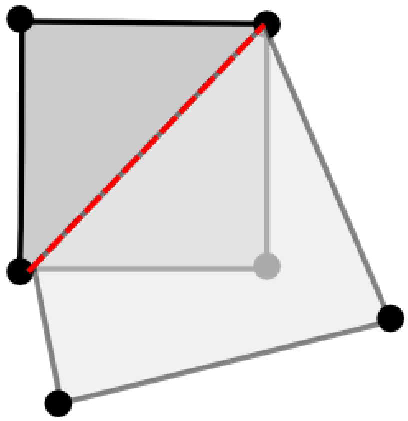
- Slightly more formal definition
  - a *cell* is a convex polytope in
  - a *proper face* of a cell is a lower dimension convex polytope subset of a cell



Cell Complexes (meshes)

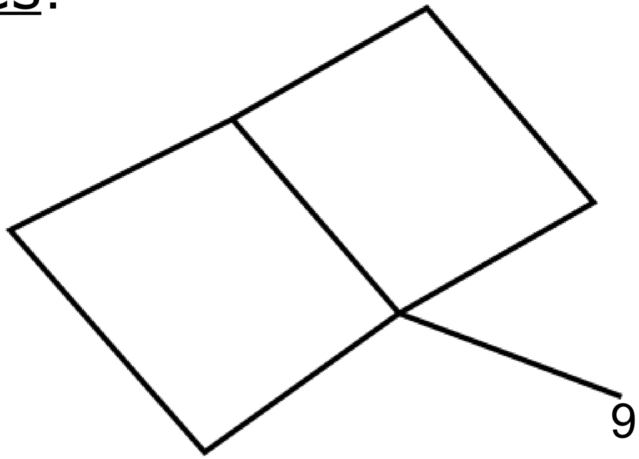
- a collection of cells is a complex **iff**
  - every face of a cell belongs to the complex
  - For every cells C and C′, their intersection either is empty or is a common face of both





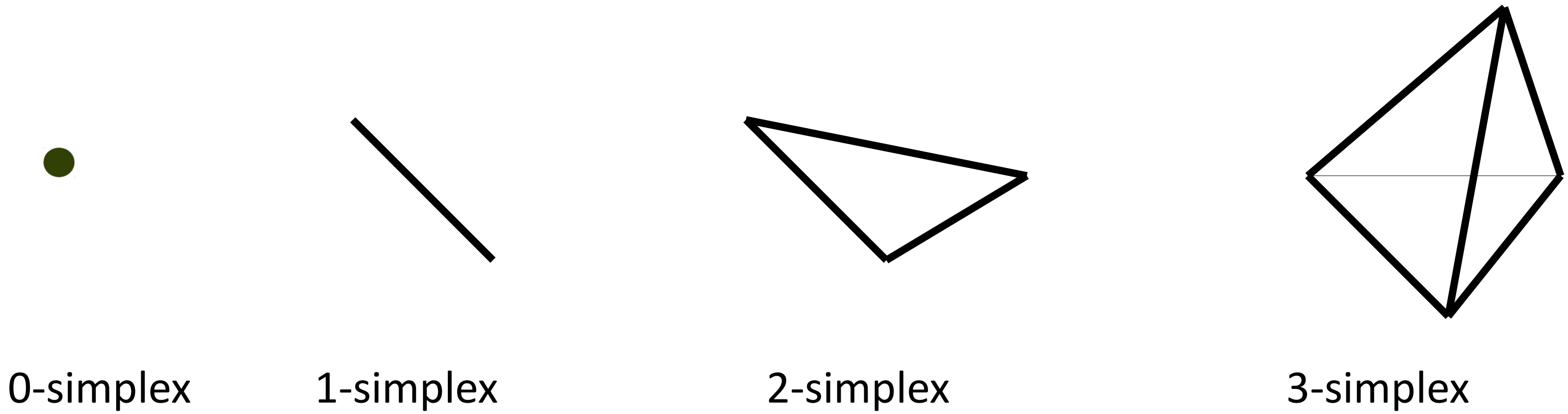
Maximal Cell Complex

- the **order** of a cell is the number of its sides (or vertices)
- a complex is a **k-complex** if the maximum of the order of its cells is *k*
- a cell is **maximal** if it is not a face of another cell
- a k-complex is **maximal** *iff* all maximal cells have order k
- short form : <u>no dangling edges</u>!



Simplicial Complex

- A cell complex is a **simplicial complex** when the cells are simplexes
- A **d-*simplex*** is the convex hull of $d+1$ points in



Sotto-Simplesso

- Un simplesso σ' è detto *faccia* di un simplesso σ se e' definito da un sottoinsieme dei vertici di σ
-
- Se $\sigma \neq \sigma'$ si dice che é una faccia propria

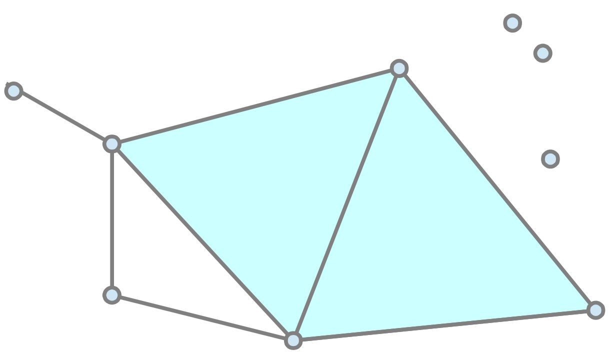
Complesso Simpliciale

- Una collezione di simplessi Σ e' un k-complesso simpliciale se:

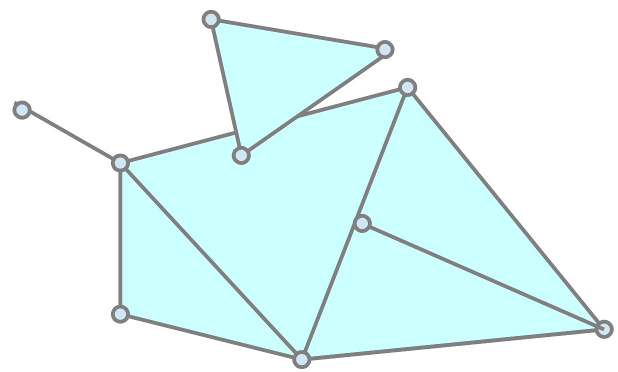
$$\forall \sigma_{1,} \sigma_2 \in \Sigma \quad \sigma_1 \cap \sigma_2 \neq \emptyset \rightarrow \sigma_1 \cap \sigma_2 \text{ is a simplex of } \Sigma$$

$$\forall \sigma \in \Sigma \text{ all the faces of } \sigma \text{ belong to } \Sigma$$

$$k \text{ is the maximum order } \forall \sigma \in \Sigma$$



OK

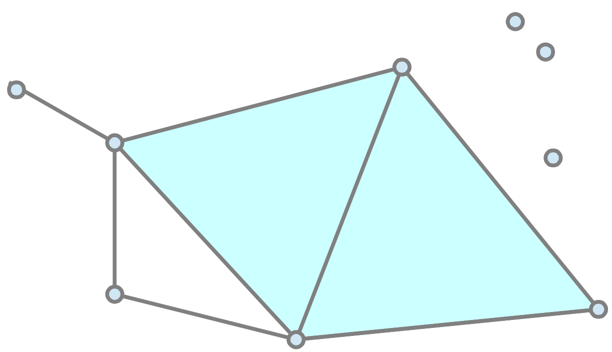


Not Ok

Complesso Simpliciale

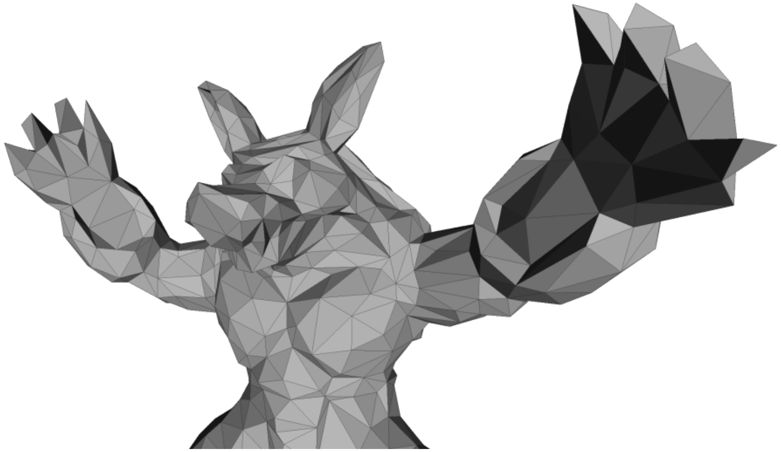
- Un simplesso σ è massimale in un complesso simpliciale Σ se non è faccia propria di nessun altro simplesso di Σ
- Un k-complesso simpliciale Σ è massimale se tutti simplessi massimali sono di ordine k
  - In pratica non penzolano pezzi di ordine inferiore

Non maximal 2-simplicial complex



Meshes, at last

- When talking of *triangle mesh* the intended meaning is a **maximal 2-simplicial complex**



Topology vs Geometry

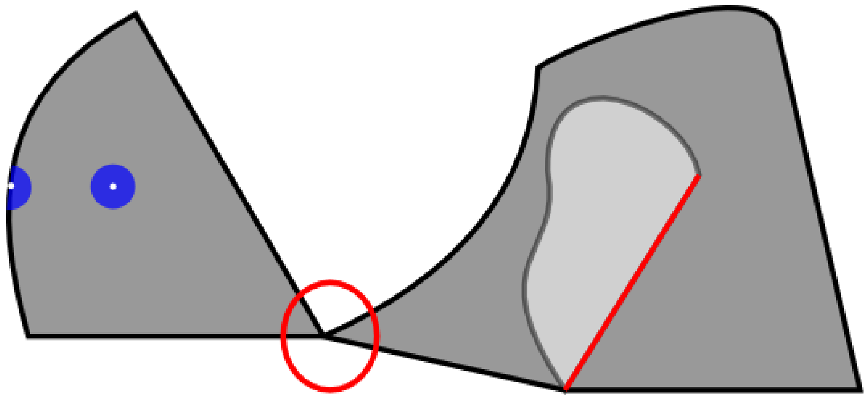
- Di un complesso simpliciale e' buona norma distinguere
  - Realizzazione geometrica
    - Dove stanno effettivamente nello spazio i vertici del nostro complesso simpliciale
  - Caratterizzazione topologica
    - Come sono connessi combinatoriamente i vari elementi

Topology vs geometry 2

- Nota: Di uno stesso oggetto e' possibile dare rappresentazioni con eguale realizzazione geometrica ma differente caratterizzazione topologica (molto differente!) Demo kleine
- Nota: Di un oggetto si puo' dire molte cose considerandone solo la componente topologica
  - Orientabilita
  - componenti connesse
  - bordi

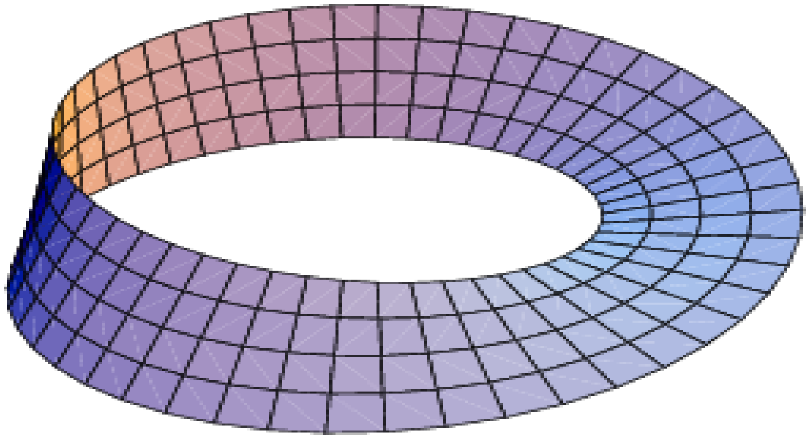
Manifoldness

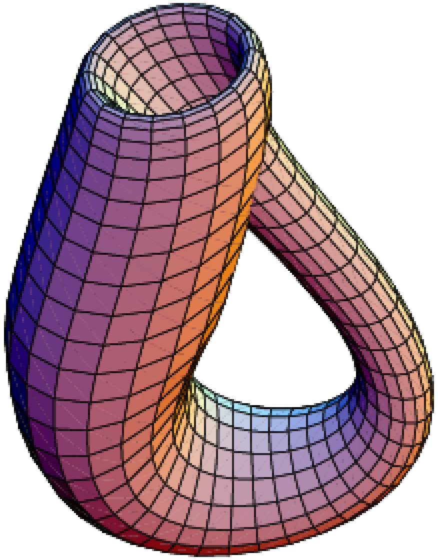
- a surface S is **2-manifold** *iff:*
  - the neighborhood of each point is homeomorphic to Euclidean space in two dimension
    or … in other words..
  - the neighborhood of each point is homeomorphic to a disk (or a semidisk if the surface has boundary)



Orientability

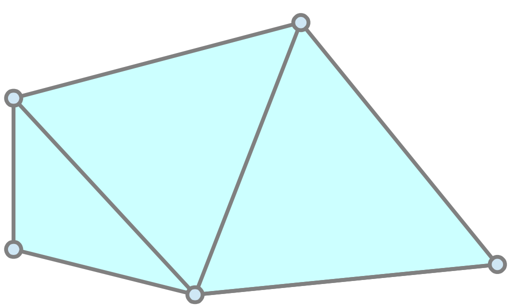
- A surface is **orientable** if it is possible to make a consistent choice for the normal vector
  - …it has two sides
- Moebius strips, klein bottles, and non manifold surfaces are not orientable





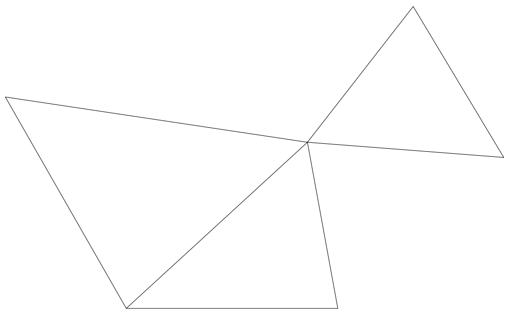
Incidenza Adiacenza

- Due simplessi σ e σ' sono incidenti se σ è una faccia propria di σ' o vale il viceversa.
- Due k-simplessi sono m-adiacenti (k>m) se esiste un m-simplesso che è una faccia propria di entrambi.
  - Due triangoli che condividono un edge sono 1 -adiacenti
  - Due triangoli che condividono un vertice sono 0-adiacenti



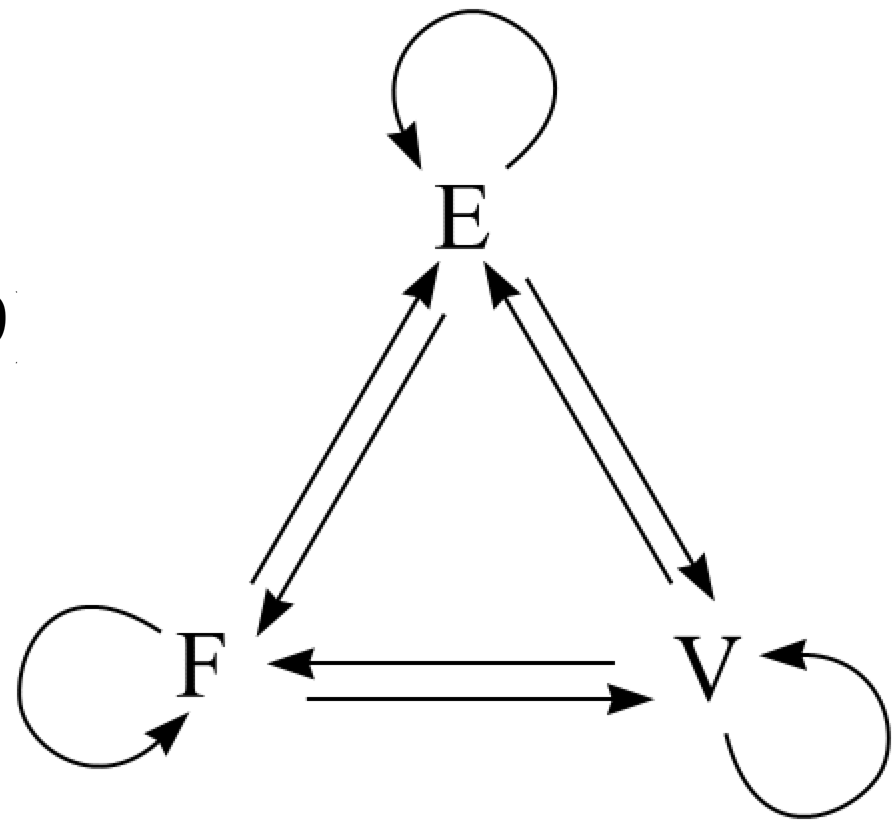
Relazioni di Adiacenza

- Per semplicità nel caso di mesh si una relazione di adiacenza con un una coppia (ordinata!) di lettere che indicano le entità coinvolte
  - FF adiacenza tra triangoli
  - FV i vertici che compongono un triangolo
  - VF i triangoli incidenti su un dato vertice



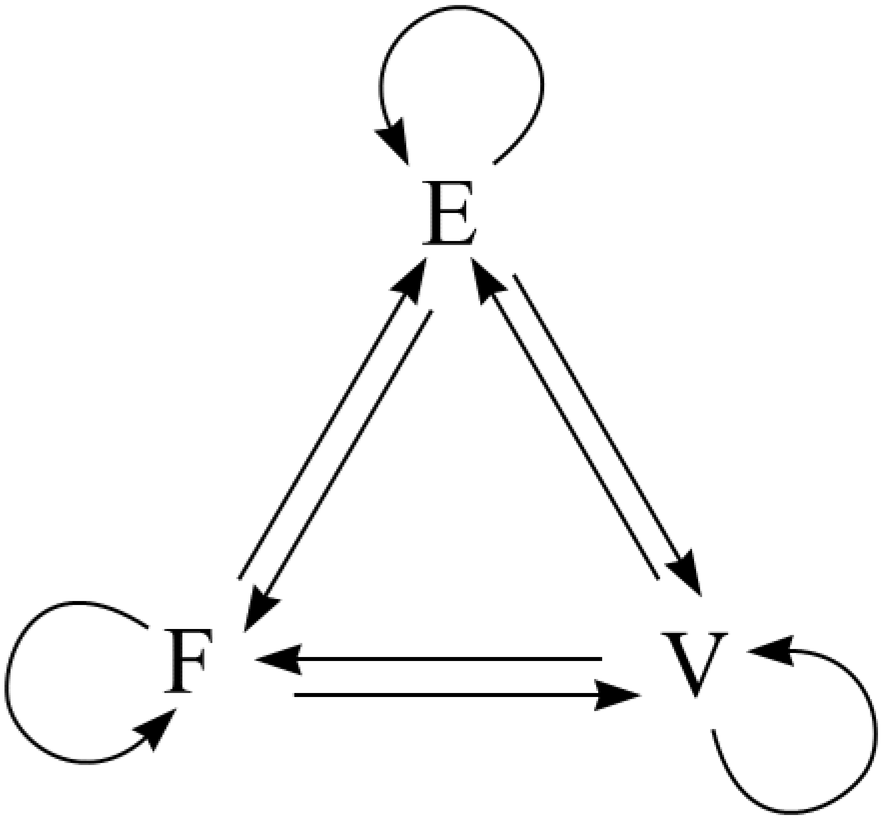
Relazioni di adiacenza

- Di tutte le possibili relazioni di adiacenza di solito vale la pena se ne considera solo un sottoinsieme (su 9
e ricavare le altre proceduralmente



Relazioni di adiacenza

- FF ~ 1-adiacenza
- EE ~ 0 adiacenza
- FE ~ sottofacce proprie di F con dim 1
- FV ~ sottofacce proprie di F con dim 0
- EV ~ sottofacce proprie di E con dim 0
- VF ~ F in Σ : V sub faccia di F
- VE ~ E in Σ : V sub faccia di E
- EF ~ F in Σ : E sub faccia di F
- VV ~ V' in Σ : Esiste E (V,V')



Partial adiacency

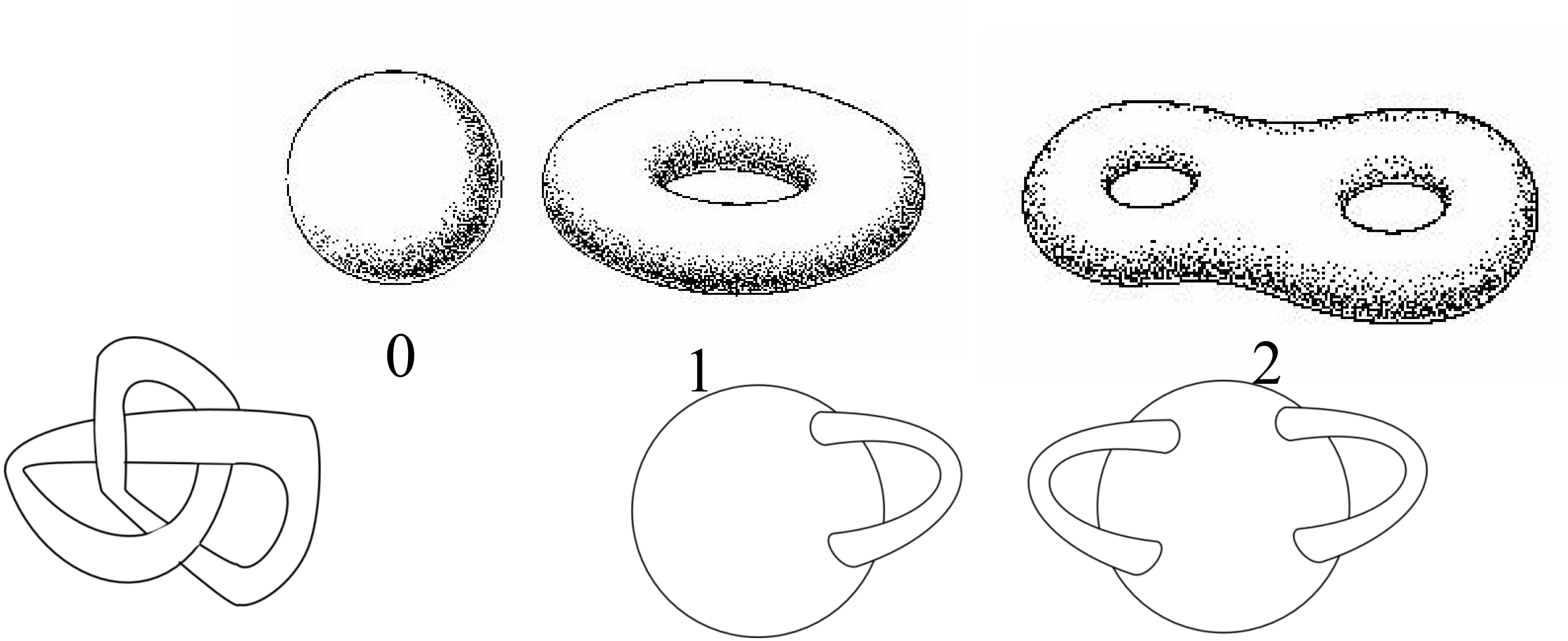
- Per risparmiare a volte si mantiene una informazione di adiacenza parziale
  - VF* memorizzo solo un riferimento dal vertice ad una delle facce e poi 'navigo' sulla mesh usando la FF per trovare le altre facce incidenti su V

Relazioni di adiacenza

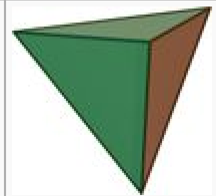
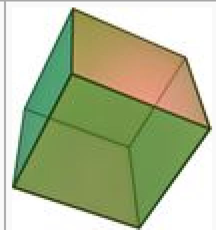
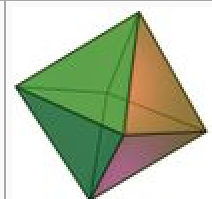
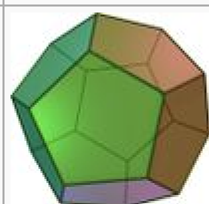
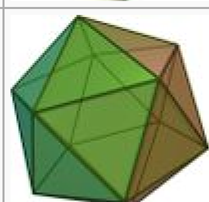
- In un 2-complesso simpliciale immerso in R3, che sia 2 manifold
  - FV FE FF EF EV sono di cardinalità bounded (costante nel caso non abbia bordi)
    - |FV|= 3 |EV| = 2 |FE| = 3
    - |FF| <= 2
    - |EF| <= 2
  - VV VE VF EE sono di card. variabile ma in stimabile in media
    - |VV|~|VE|~|VF|~6
    - |EE|~10
    - F ~ 2V

Genus

❖ The **Genus** of a closed surface, orientable and 2-manifold is the maximum number of cuts we can make along non intersecting closed curves without splitting the surface in two.

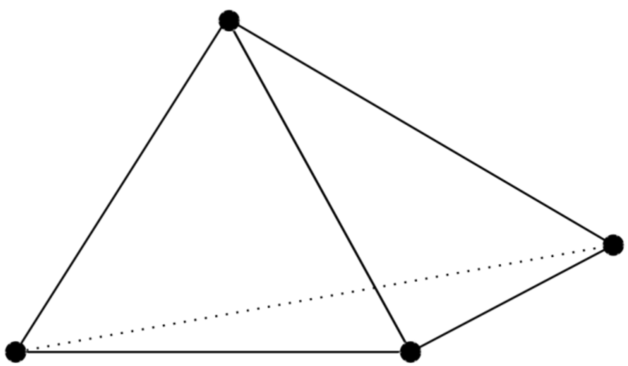


❖ …also known as the number of *handles*

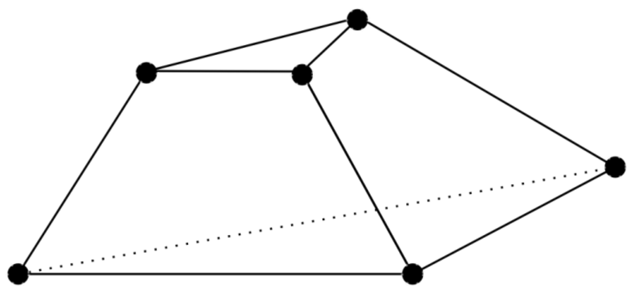
Name	Image	*V* (vertices)	*E* (edges)	*F* (faces)	Euler characteristic: *V* − *E* + *F*
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Euler characteristics

- χ = 2 for any *simply connected* polyhedron
- proof by construction…
- play with examples:



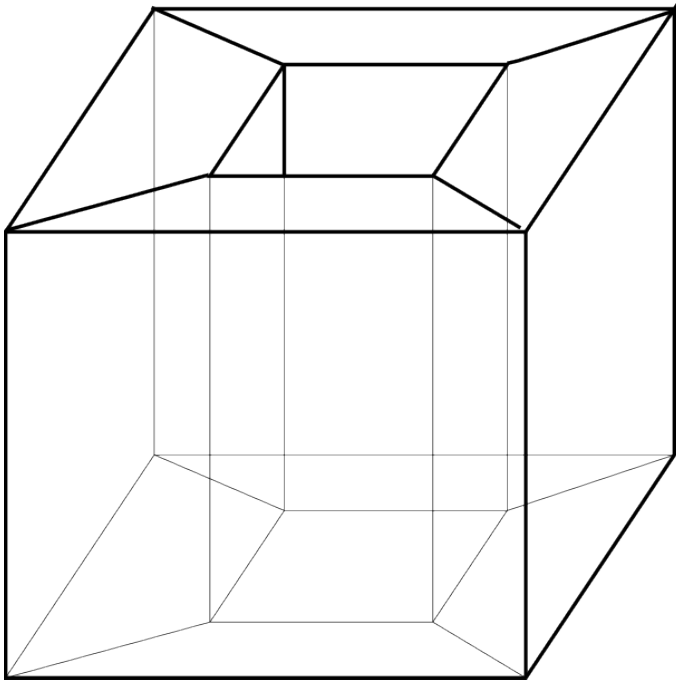
$$\chi = V - E + F$$
$$\chi = 4 - 6 + 4 = 2$$



$$\chi = (V + 2) - (E + 3) + (F + 1) =$$
$$\chi = (4 + 2) - (6 + 3) + (4 + 1) = 2$$

Euler characteristics

- let's try a more complex figure…



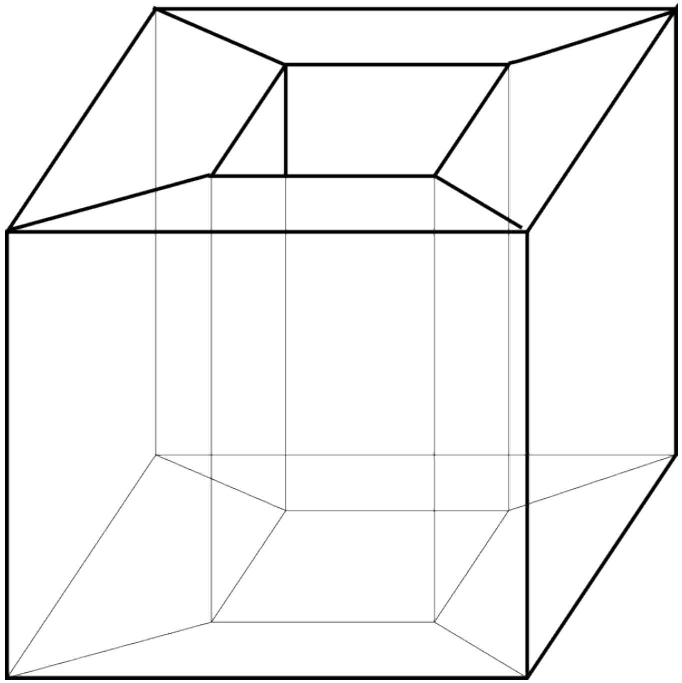
$$\chi = V - E + F$$
$$\chi = 16 - 32 + 16 = \mathbf{0}$$

- why =0 ?

Euler characteristics

$$\chi = 2 - 2g$$

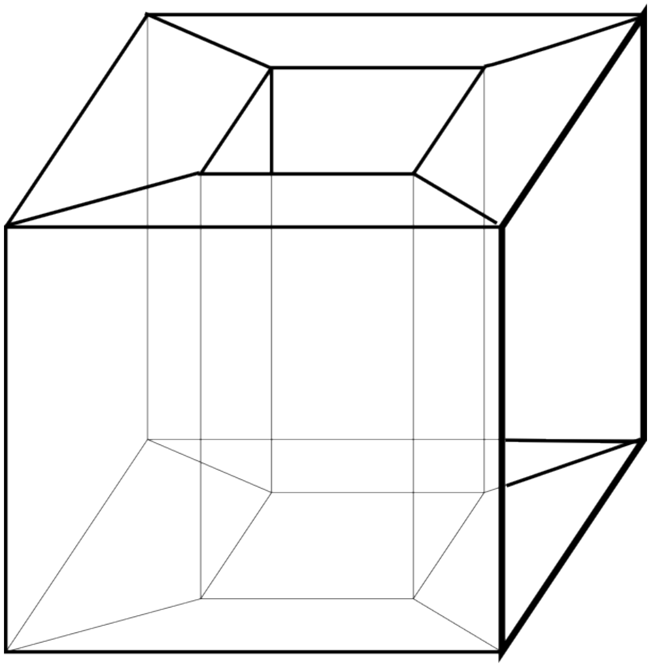
- where *g* is the genus of the surface



$$\chi = V - E + F$$
$$\chi = 16 - 32 + 16 = 0 = \boldsymbol{2 - 2g}$$

Euler characteristics

- let's try a more complex figure…remove a face. The surface is not closed anymore



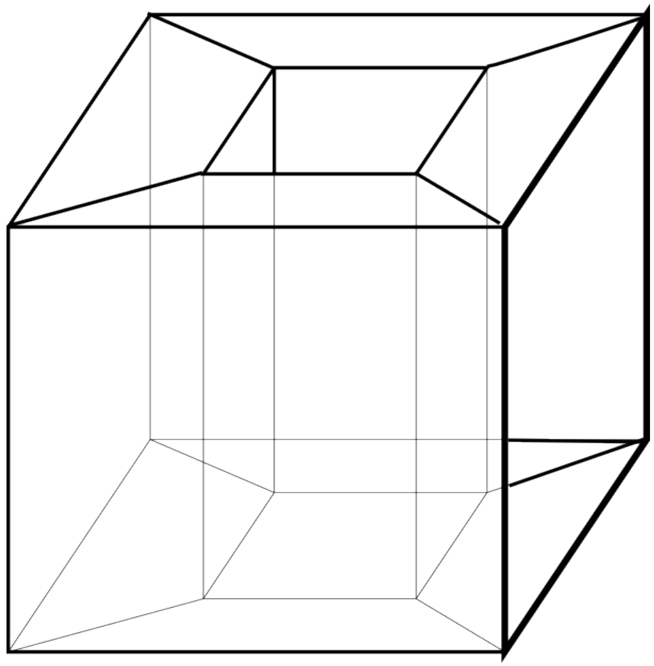
$$\chi = V - E + F$$
$$\chi = 16 - 32 + 15 = -\mathbf{1}$$

- why =-1 ?

Euler characteristics

$$\chi = 2 - 2g - b$$

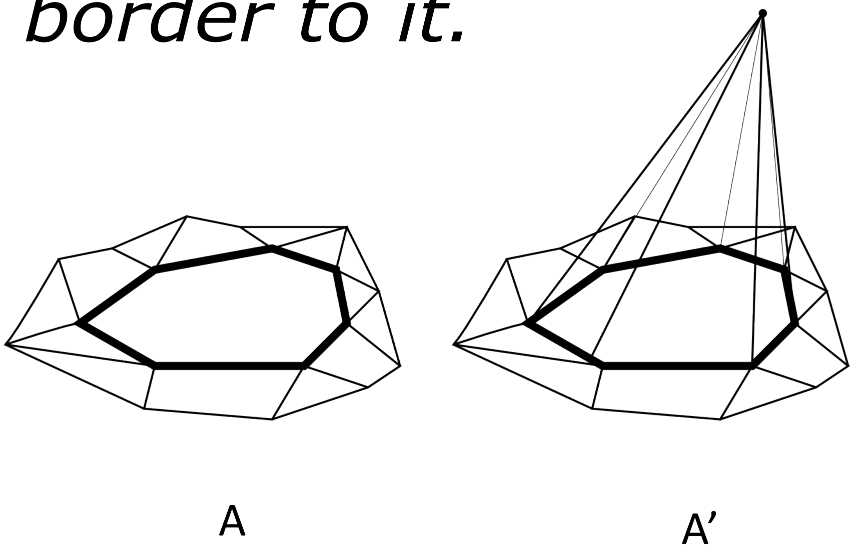
- where *b* is the number of borders of the surface



$$\chi = V - E + F$$
$$\chi = 16 - 32 + 15 = -1 = 2 - 2g - b$$

Euler characteristics

- *Remove the border by adding a new vertex and connecting all the **k** vertices on the border to it.*



$$X' = X + V' - E' + F' = X + 1 - k + k = X + 1$$

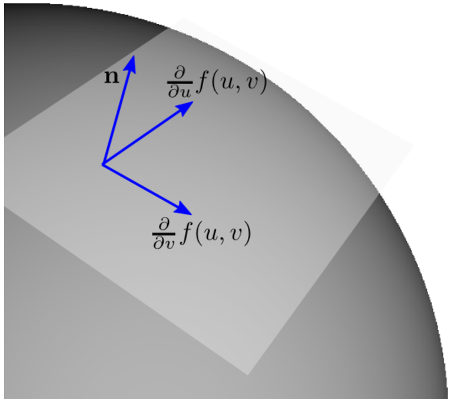
Differential quantities:normals

- The (unit) **normal** to a point is the (unit) vector perpendicular to the tangent plane

$$\text{implicit surface}: f(x, y, z) = 0$$

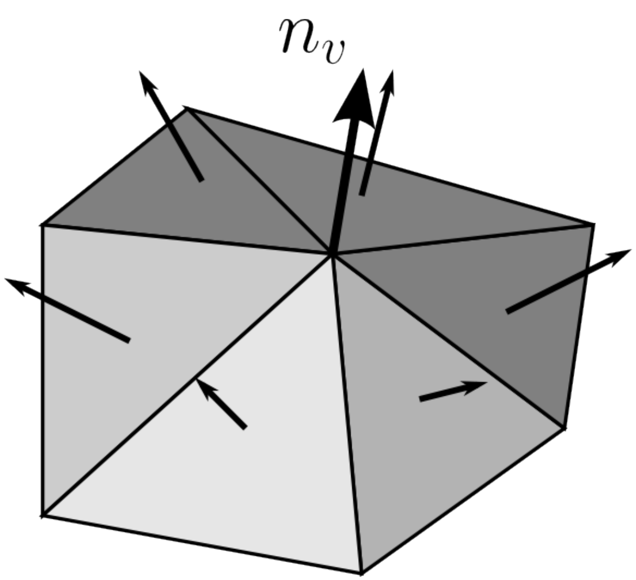
$$n = \frac{\nabla f}{[\nabla f]}$$

$$\text{parametric surface}:$$
$$f(u, v) = (f_x(u, v), f_y(u, v), f_z(u, v))$$
$$n = \frac{\partial}{\partial u} f \times \frac{\partial}{\partial v} f$$



Normals on triangle meshes

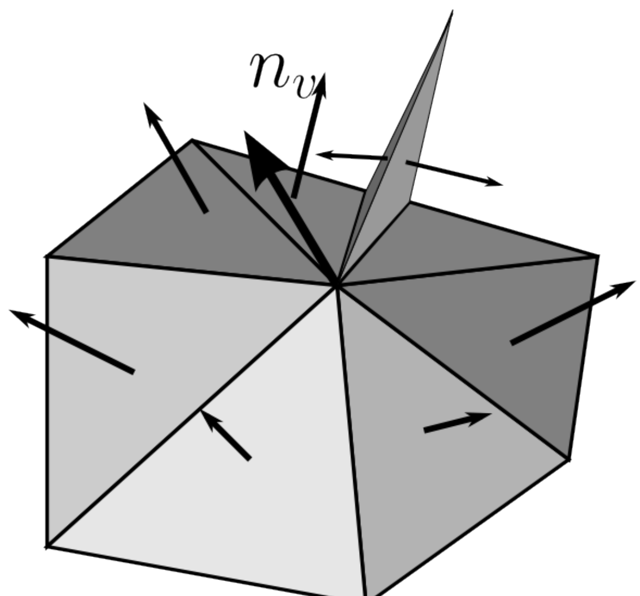
- Computed per-vertex and interpolated over the faces
- Common: consider the tangent plane as the average among the planes containing all the faces incident on the vertex



$$n_v = \frac{1}{\#N(v)} \sum_{f \in N(v)} n_f$$
$$N(v) = \{f : f \; coface \; of \; v\}$$

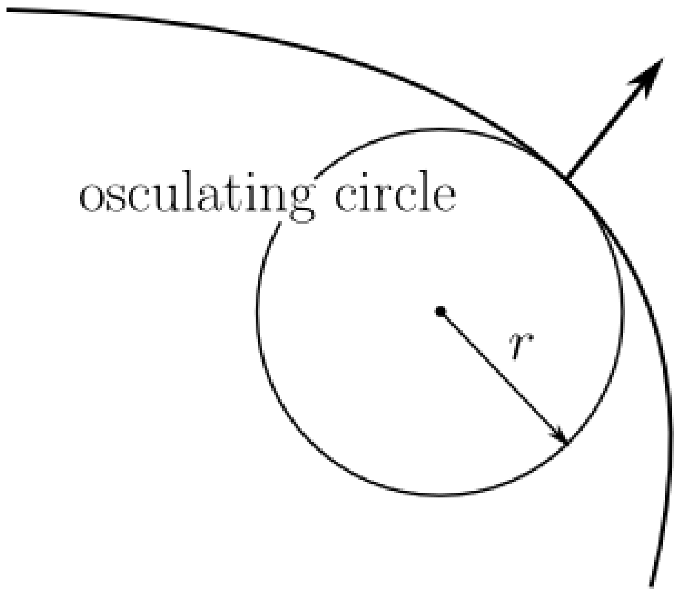
Normals on triangle meshes

- Does it work? Yes, for a “good” tessellation
  - Small triangles may change the result dramatically
  - Weighting by edge length / area / angle helps



Differential quantities: Curvature

- The curvature is a measure of how much a line is curve



r: radius or curvature

$\kappa = \frac{1}{r}$: curvature

$x(t), y(t)$ *a parametric curve*

ϕ *tangential angle*

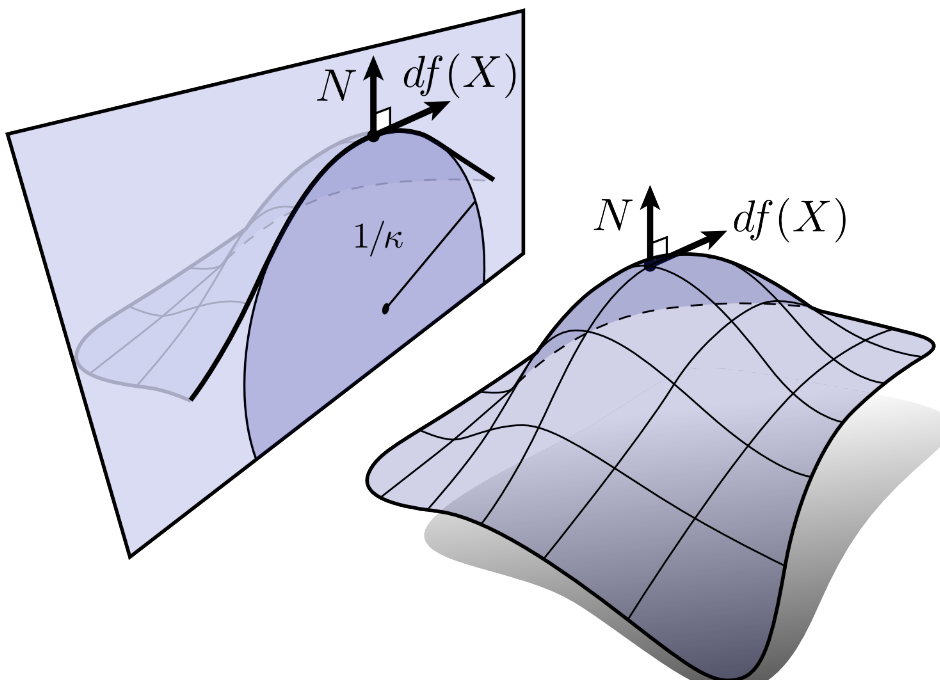
%s arc lenght

$$\kappa \equiv \frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \frac{d\phi/dt}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} = \frac{d\phi/dt}{\sqrt{x'^2 + y'^2}}$$

$$\kappa = \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}}$$

Curvature on a surface

- Given the normal at point *p* and a tangent direction θ
- The curvature along θ is the 2D curvature of the intersection between the plane and the surface

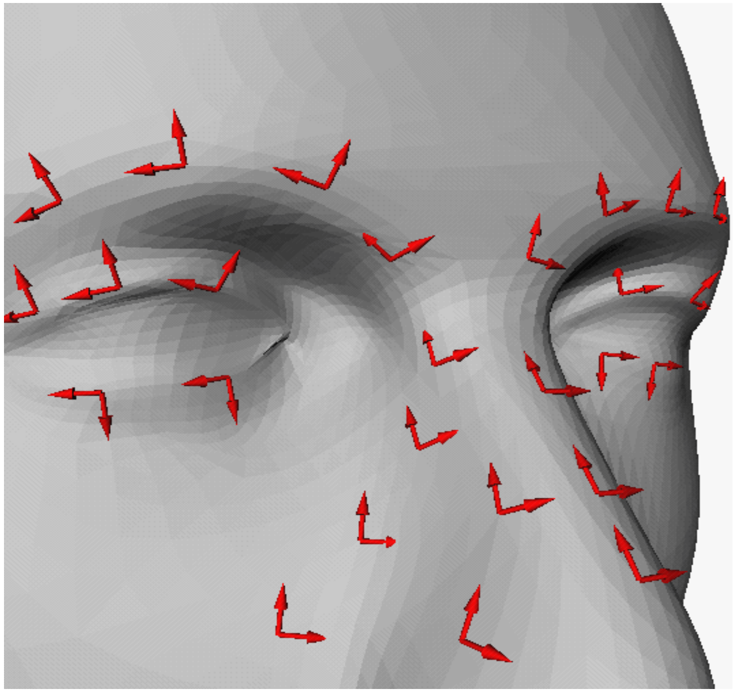


Curvatures

- A curvature for each direction
- Take the two directions for which curvature is max and min

$$\kappa_1, \kappa_2 \quad principal \quad curvatures$$
$$e_1, e_2 \quad principal \quad directions$$

- the directions of max and min curvature are orthogonal



[Meyer02]

Gaussian and Mean curvature

- Gaussian curvature: the product of principal curvatures

$$\kappa_G \equiv K \equiv \kappa_1 \cdot \kappa_2$$

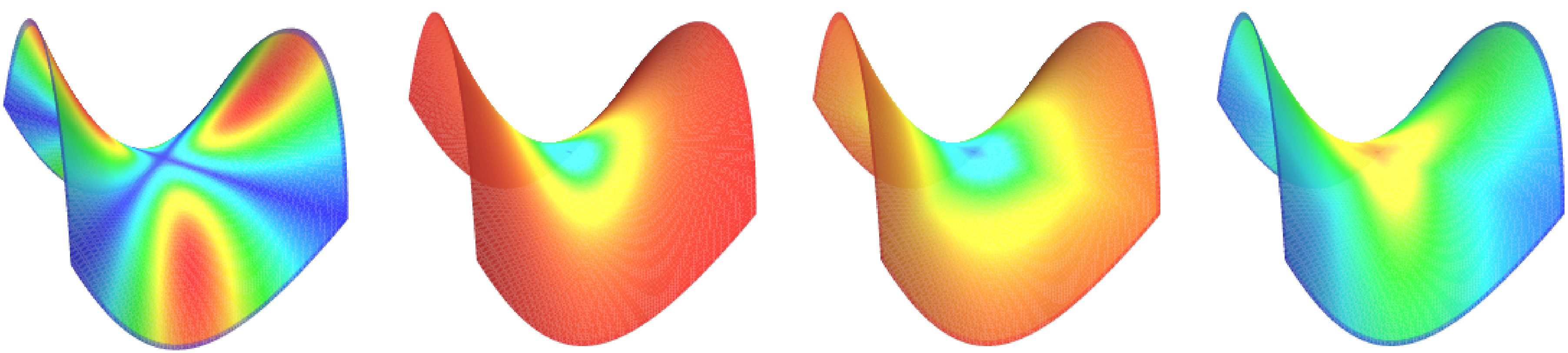
- Mean curvature: the average of principal curvatures

$$\bar{\kappa} \equiv H \equiv \frac{\kappa_1 + \kappa_2}{2}$$

Examples..

- Red:low → red:high (not in the same scale)

mean gaussian min max



[Meyer02]

Gaussian Curvature

- Gaussian curvature is an intrinsic property
  - It can be computed by a bidimensional inhabitant of the surface by walking around a fixed point **p** and keeping at a distance *r.*

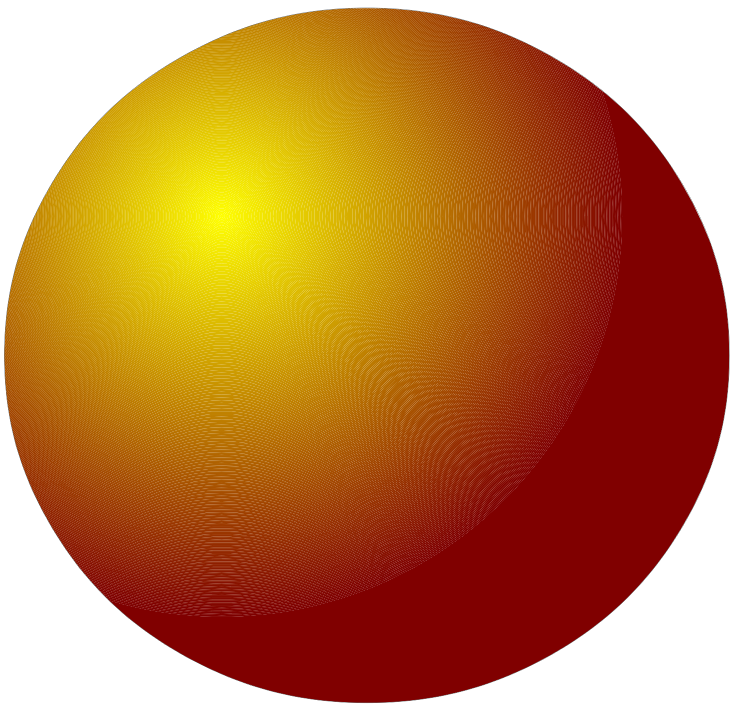
$$\kappa_G = \lim_{r \to 0} \left(2\pi r - C\left(r\right)\right) \cdot \frac{3}{\pi r^3}$$

with *C(r)* the distance walked

- Delevopable surfaces: surfaces whose Gaussian curvature is 0 everywhere

Gaussian Curvature

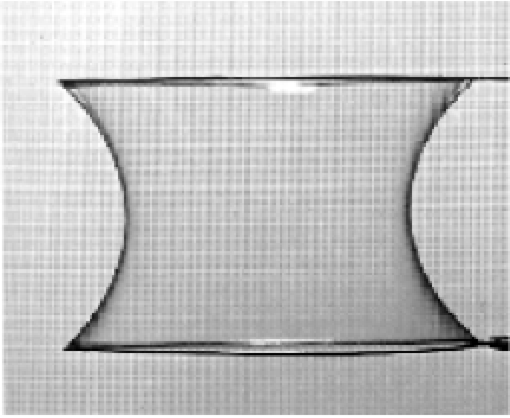
$\kappa_G > 0$



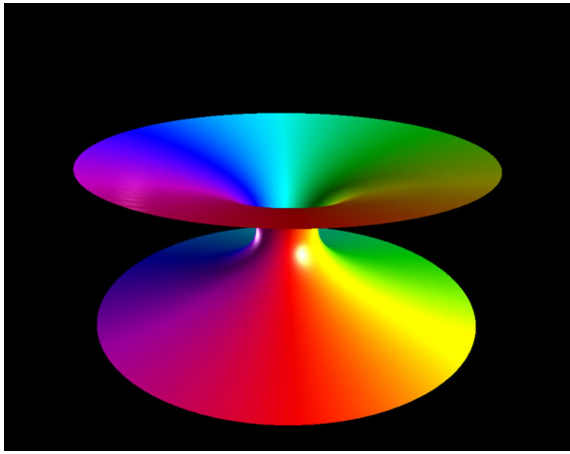
$\kappa_G < 0$

Mean Curvature

- Divergence of the surface normal
  - divergence is an operator that measures a vector field's tendency to originate from or converge upon a given point



- Minimal surface and minimal area surfaces
  - A surface is *minimal* when its mean curvature is 0 everywhere
  - All minimal area surfaces have mean curvature 0
- The surface tension of an interface, like a soap bubble, is proportional to its mean curvature



Mean Curvature

- Let A be the area of a disk around *p*. The mean curvature

$$2\bar{\kappa}\, n = \lim_{diam(A)\to 0} \frac{\nabla A}{A}$$

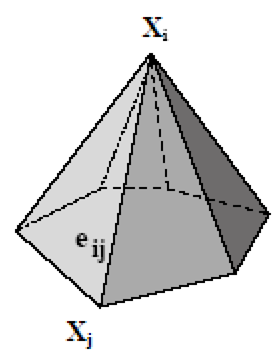
- the mean derivative is (twice) the divergence of the normal

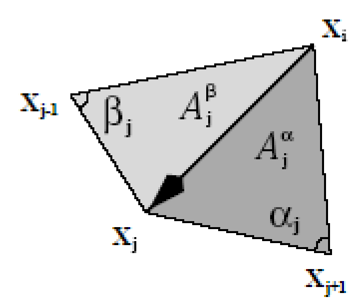
$$2\bar{\kappa} = \nabla \cdot n = \frac{\partial}{\partial x} n + \frac{\partial}{\partial y} n + \frac{\partial}{\partial z} n$$

Mean curvature on a triangle mesh

$$H(p)=\frac{1}{2A}\sum_{i=1}^{n}\left(\cot\alpha_i+\cot\beta_i\right)\left\|p-p_i\right\|$$

where α_j and β_j are the two angles opposite to the edge in the two triangles having the edge e_{ij} in common
A is the sum of the areas of the triangles





Gaussian curvature on a triangle mesh

- It's the *angle defect* over the area

$$\kappa_G(v_i) = \frac{1}{3A}\left(2\pi - \sum_{t_j\, adj\, v_i} \theta_j\right)$$

- **Gauss-Bonnet Theorem:** The integral of the Gaussian Curvature on a closed surface depends on the Euler number

$$\int_S \kappa_G = 2\pi\chi$$

Example data structure

- Simplest
- List of triangles:
  - For each triangle store its coords.
  -
  - 1. (3,-2,5), (3,6,2), (-6,2,4)
  - 2. (2,2,4), (0,-1,-2), (9,4,0)
  - 3. (1,2,-2), (3,6,2), (-4,-5,1)
  - 4. (-8,2,7), (-2,3,9), (1,2,-2)
- How to find any adjacency?
- Does it store FV?

Example data structure

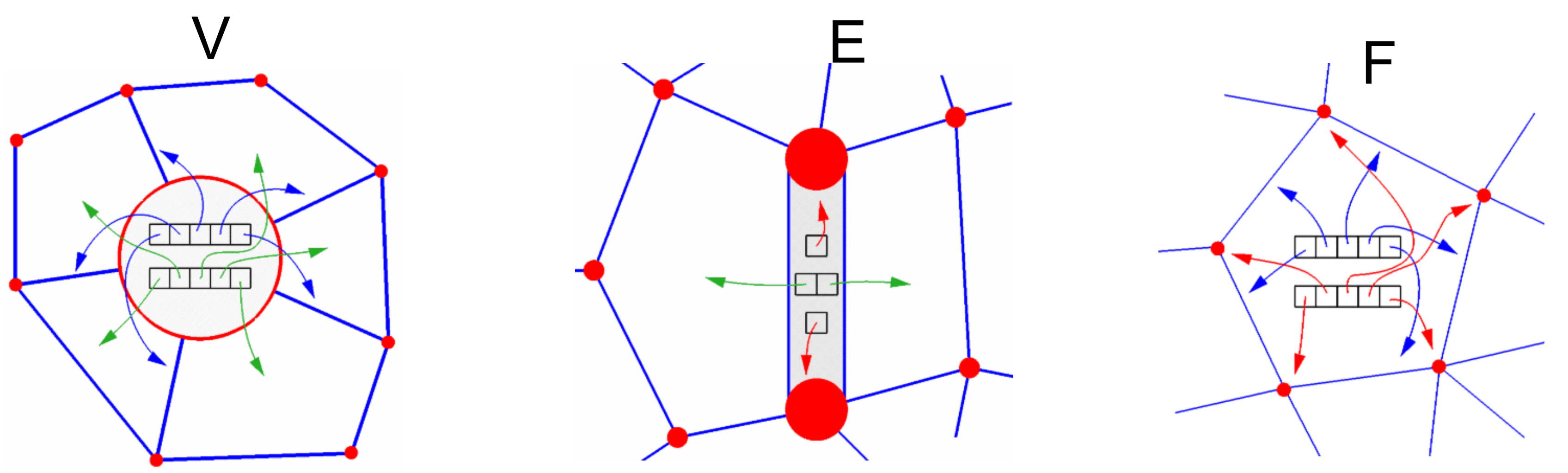
- Slightly better
- List of unique vertices with indexed faces
  - Storing the FV relation

- Vertices:
  1. (-1.0, -1.0, -1.0)
  2. (-1.0, -1.0, 1.0)
  3. (-1.0, 1.0, -1.0)
  4. (-1, 1, 1.0)
  5. (1.0, -1.0, -1.0)
  6. (1.0, -1.0, 1.0)
  7. (1.0, 1.0, -1.0)
  8. (1.0, 1.0, 1.0)

- Faces:
  1. 1 2 4
  2. 5 7 6
  3. 1 5 2
  4. 3 4 7
  5. 1 7 5

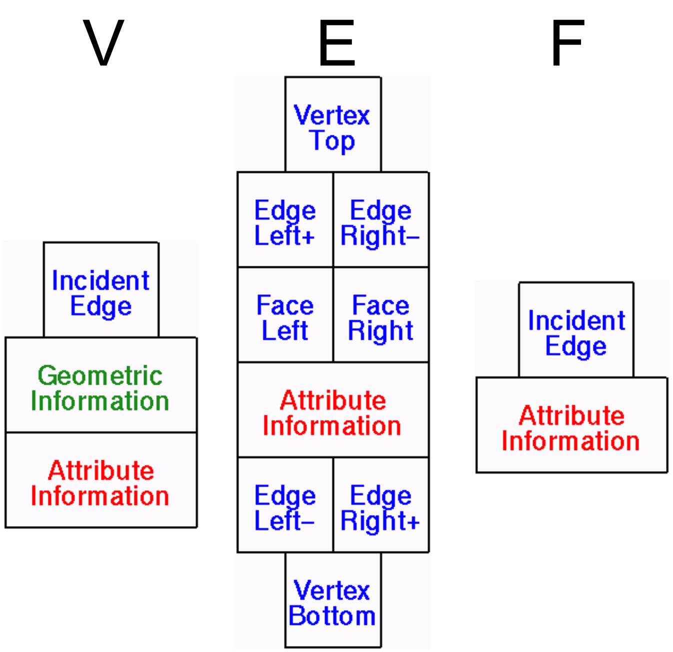
Example data structure

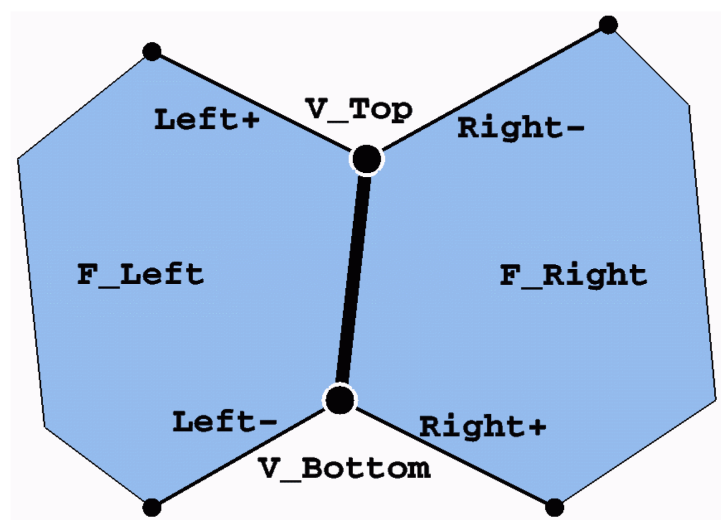
- Issue of Adjacency
  - Vertex, Edge, and Face Structures
    - Each element has list of pointers to all incident elements
    - Queries depend only on local complexity of mesh!
    - Slow! Big! Too much work to maintain!
    - Data structures do not have fixed size



Example data structure

- Winged edge
  - Classical real smart structure
  - Nice for generic polygonal meshes
  - Used in many sw packages





Winged Edge

- Winged edge
  - Compact
  - All the query requires some kind of "traversal"
  - Not fitted for rendering...